Notes on Dynamic Programming:

The Basic Problem

- let X be a Metric Space, Consider the problem: given $x_0 \in X$, find a sequence $(x_t)_{t=1}^{\infty} \in X^{\infty}$, s.t.:

$$x_t = \underset{\substack{x_{t+1} \in \Gamma(x_t) \\ \forall t = 0,1,\dots}}{\operatorname{argmax}} \sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1})$$

- X is State Space, x_0 is initial State.
- $\Gamma : X \rightrightarrows X$ transition correspondence
- Given x_0, any seq $\underbrace{(x_t)_{t=1}^{\infty}}_{\text{feasible plan}}$ s.t. $x_{t+1} \in \Gamma(x_t)\ \forall t$
- $u : X \times X \longmapsto \mathbb{R}$ is the 1-period return fcn.
- $\delta \in (0,1)$ discount factor.

Assumption $\forall (x_t)_{t=1}^{\infty}$ s.t. $x_{t+1} \in \Gamma(x_t)\ \forall t$:

$$\lim_{T \to \infty} \sum_{t=0}^{T} \delta^t u(x_t, x_{t+1}) \quad \exists \text{ (possibly infinite)}$$

[bdd u sufficient]

$(X, x_0, \Gamma, u, \delta)$ is the standard dynamic prog. problem.

let $D(X, \Gamma, u, \delta)$ denote the class of problems:

$$:= \{(X, x_0, \Gamma, u, \delta) : x_0 \in X\}$$

- Defn. $\Omega(x)$ the set of feasible plans

$$:= \{(x_t)_{t=1}^{\infty} \in X^{\infty} : x_1 \in \Gamma(x) \text{ and } x_{t+1} \in \Gamma(x_t), \forall t \in \mathbb{N}\}$$

- Defn. $U((x_t), x)$ value of objective fcn given $x_0 \in X$ and $(x_t) \in \Omega(x)$.

$$:= u(x, x_1) + \sum_{t=1}^{\infty} \delta^t u(x_t, x_{t+1})$$

- Defn. Value fcns $V(x) := \sup_{(x_t) \in \Omega(x)} \{U((x_t), x)\}$

If solution $\exists\ \forall x$, sup $\Leftrightarrow$ max.

THM: (Bellman)

For $D(X, \Gamma, u, \delta)$, $\forall x_0 \in X$, and $(x_t^*) \in \Omega(x_0)$

If: $V(x_0) = U(\{x_t^*\}, x_0)$,

then: $V(x_0) = u(x_0, x_1^*) + \delta V(x_1^*)$

$V(x_t^*) = u(x_t^*, x_{t+1}^*) + \delta V(x_{t+1}^*)$

If u is bdd, the converse holds.

Pf: ($\Rightarrow$) since (x_t^*) is optimal path:

$$\forall (x_t) \in \Omega(x_0),\ u(x_0, x_1^*) + \sum_{t=1}^{\infty} \delta^t u(x_t^*, x_{t+1}^*) \geq u(x_0, x_1) + \sum_{t=1}^{\infty} \delta^t u(x_t, x_{t+1})$$

Consider any $(x_2, x_3, \cdots) \in \Omega(x_1^*)$

Then $(x_1^*, x_2, x_3, \cdots) \in \Omega(x_0)$ since $x_1^* \in \Gamma(x_0)$

$$\Rightarrow \cancel{u(x_0, x_1^*)} + \sum_{t=1}^{\infty} \delta^t u(x_t^*, x_{t+1}^*) \geq \cancel{u(x_0, x_1^*)} + \delta u(x_1^*, x_2) + \sum_{t=2}^{\infty} \delta^t u(x_t, x_{t+1})$$

divide δ $\Rightarrow$ $u(x_1^*, x_2^*) + \sum_{t=2}^{\infty} \delta^{t-1} u(x_t^*, x_{t+1}^*)$

$$\geq u(x_1^*, x_2) + \sum_{t=2}^{\infty} \delta^{t-1} u(x_t, x_{t+1})$$

$$\Rightarrow U((x_2^*, x_3^*, \cdots), x_1^*) \geq U((x_2, x_3, \cdots), x_1^*)$$

$$\forall (x_2, x_3 \cdots) \in \Omega(x_1^*)$$

$$\Rightarrow V(x_1^*) = U((x_2^*, x_3^*, \cdots), x_1^*)$$

By Hypo: $V(x_0) = U((x_t^*), x_0) = u(x_0, x_1^*) + \delta V(x_1^*)$

Continue by induction ....

($\Leftarrow$) u is bdd:

$V(x_0) = u(x_0, x_1^*) + \delta V(x_1^*)$

$V(x_t^*) = u(x_t^*, x_{t+1}^*) + \delta V(x_{t+1}^*)$

$$\Rightarrow V(x_0) = u(x_0, x_1^*) + \delta V(x_1^*)$$

$$= u(x_0, x_1^*) + \delta u(x_1^*, x_2^*) + \delta^2 V(x_2^*)$$

$$= u(x_0, x_1) + \sum_{t=1}^{T} \delta^t u(x_t^*, x_{t+1}^*) + \boxed{\delta^{T+1} V(x_{T+1}^*)} \to 0 \quad (u \text{ bdd}, V \text{ bdd})$$

$$\Rightarrow V(x_0) = U((x_t^*)_{t=1}^{\infty}, x_0)$$

- The optimal Policy correspondence

Suppose $D(X, \Gamma, u, \delta)$ has a solution $\forall x_0 \in X$.

$v(x_0)$ value fcn.

<u>Defn</u>: The optimal policy correspondence $P: X \rightrightarrows X$:

$$P(x_0) = \underset{y \in \Gamma(x)}{\arg\max} \{u(x_0, y) + \delta v(x_0, y)\}$$

(verify) If u is bdd, then $(x_t) \in X^\infty$ is a solution to $D(X, \Gamma, u, \delta)$ iff:

$$x_{t+1} \in P(x_t) \quad \forall t \in \mathbb{N}.$$

<u>Thm.</u> Consider any class of problem $D(X, \Gamma, u, \delta)$, $\forall w \in \mathcal{B}(X)$.

If $w(x) = \max_{y \in \Gamma(x)} \{u(x, y) + \delta w(y)\} \quad \forall x \in X$

Then $w(x) = \max_{(x_t) \in \Omega(x)} \{U((x_t), x)\} \quad \forall x \in X$.

<u>Pf</u>: Suppose $w(x) = \max_{y \in \Gamma(x)} \{u(x, y) + \delta w(y)\} \quad \forall x \in X$.

let $x \in X$, $(x_t) \in \Omega(x)$ Then

$$w(x) \geq u(x, x_1) + \delta w(x_1).$$

$$\geq u(x, x_1) + \delta u(x_1, x_2) + \delta^2(x_2)$$

$$\geq u(x, x_1) + \sum_{t=1}^{T} \delta^T u(x_t, x_{t+1}) + \delta^{T+1} w(x_{T+1}) \quad \forall T$$

$$\Rightarrow \quad w(x) \geq \sum_{t=0}^{\infty} \delta^t u(x_t, x_{t+1}) + \lim_{T \to 0} \delta^{T+1} w(x_{T+1})$$

Since $w \in \mathcal{B}(X)$, $\Rightarrow w(x) \geq U((x_t), x)$

WT construct (x_t^*) s.t. $w(x) = U((x_t^*), x)$.

$\Rightarrow$ supremum is $U((x_t^*), x)$.

choose $(x_t^*) \in \Omega(x)$ s.t.

$$w(x) = u(x, x_1^*) + \delta w(x_1^*)$$

and $\forall t \in \mathbb{N} \cap \{0\}$, $w(x_t^*) = u(x_t^*, x_{t+1}^*) + \delta w(x_{t+1}^*)$

$\forall T \in \mathbb{N} \cap \{0\}$. $w(x) = \sum_{t=0}^{T} \delta^t u(x_t^*, x_{t+1}^*) + \delta^{T+1} w(x_{T+1}^*)$.

$T \to \infty \quad \Rightarrow \quad w(x) = U((x_t^*), x) \geq U((x_t), x) \quad \mathcal{B}(X)$

$\forall (x_t) \in \Omega(x)$

Example: Optimal Growth:

Capital at start of period t is x_t, production (included depreciated capital) is $f(x_t)$, consumption is C_t, so.

$$x_{t+1} = f(x_t) - C_t.$$

Given $x_0 \geq 0$. Social planner choose $(x_t)_{t=1}^{\infty}$ ∋.

$$\max_{(x_t)_{t=1}^{\infty}} \sum_{t=0}^{\infty} \delta^t u(C_t) = \sum_{t=0}^{\infty} \delta^t u(f(x_t) - x_{t+1})$$

$$\text{s.t.} \quad 0 \leq x_{t+1} \leq f(x_t)$$

Given $u: \mathbb{R}_+ \to \mathbb{R}$ and $\delta \in (0,1)$, this is a standard DP problem with $x \in X \subset \mathbb{R}$ and transition correspondence $\Gamma(x) = [0, f(x)]$

Existence & uniqueness

- Existence
  - u is continuous and bdd
  - Γ is compact-valued and continuous
  - $CB(X)$ denote set of conti. bdd. real valued fcns on X. (complete)
  - Defn: $T: CB(X) \longmapsto CB(X)$

$$T(W)(x) := \max_{y \in \Gamma(x)} \{u(x,y) + \delta W(y)\} \quad \forall x \in X.$$

- $T(W)(x)$ is conti by Thm. of max.
  [• $\Gamma(x)$ compact-valued, cont.
  • u cont.]
- A fixed pt of T is a value fcn by principal of optimality

$$T(V)(x) = V(x)$$

Claim: $\exists$ a unique and global stable fixed pt of T.

Check that $T(\cdot)(x)$ sats. Blackwell condi.:
- (monotonicity) $W \geq W' \Rightarrow T(W) \geq T(W')$
- (discounting) $\exists \delta' \in (0,1)$ s.t. $T(W+\alpha) \leq T(W) + \delta' \alpha \quad \forall \alpha \in \mathbb{R}_+$.

const. fcn.

Then $T(\cdot)(x)$ is a contraction, i.e. $\exists \beta \in [0,1)$ ∋. $d(T(x), T(y)) \leq \beta d(x,y)$
$T: X \mapsto X$

Since $CB(X)$ is complete, by contraction mapping thm
$\Rightarrow \exists V \in CB(X)$ unique s.t. $T(V)(x) = V(x)$. ///

Single valueness of opt. policy correspondence

Thm of maximum $\Rightarrow$ $P(x)$ is compact-valued and uhc.

? singleton

- $X \neq \emptyset$, convex, $\subset \mathbb{R}^n$ $(n \in \mathbb{N})$
- $Gr(\Gamma)$ is convex:

  $\forall x, x' \in X$, $y \in \Gamma(x)$, $y' \in \Gamma(x')$, $\lambda \in (0,1)$:

  $$\lambda y + (1-\lambda) y' \in \Gamma(\lambda x + (1-\lambda) x')$$

- u is strictly concave on $Gr(\Gamma)$:

  $\forall (x, y) \neq (x', y') \in Gr(\Gamma)$, $\lambda \in (0,1)$

  $$u(\lambda (x,y) + (1-\lambda)(x', y')) > \lambda u(x,y) + (1-\lambda) u(x', y')$$

  $\Rightarrow$ Then $P(x)$ is singleton.

Eg: $x_0 \geq 0$, planner's problem:

$$\max_{\{x_t\}_{t=1}^{\infty}} \sum_{t=0}^{\infty} \delta^t u(f(x_t) - x_{t+1})$$

$$\text{s.t. } x_{t+1} \in \Gamma(x_t) := [0, f(x_t)].$$

Assuming (Inada):

- u and f are bdd and twice-diffable
- $u' > 0$, $u'' < 0$, $u'_+(0) = \infty$
- $f(0) = 0$, $f' > 0$, $f'' < 0$, $f'_+(0) = \infty$

Then: • Γ compact-valued and conti.

• $u(x_t, x_{t+1})$ is cont. in x_t, x_{t+1}

$\exists$ unique $V(x)$ s.t.

$$V(x) = \max_{0 \leq y \leq f(x)} \{ u(f(x), y) + \delta V(y) \} \quad \forall x \in \mathbb{R}_+$$

(check) • $V(\cdot)$ is strictly $\uparrow$.

• $Gr(\Gamma)$ convex.

• $u(\cdot,\cdot)$ strictly concave on $Gr(\Gamma)$

$\Rightarrow$ $P(x)$ is singleton.

• by Thm of max $\Rightarrow$ (uhc) $\Rightarrow$ cont.

(check): • $P(x)$ is strictly $\uparrow$

$\Rightarrow$ • optimal plan is <u>monotonic</u>

$$(x_t^*)_{t=1}^{\infty} = (P(x_{t-1}^*))_{t=0}^{\infty}$$

• $f(\cdot)$ is bdd $\Rightarrow$ <u>$x_{t+1}^* \leq f(x_t^*) < \infty$</u> $\forall t$,

Then $(x_t^*)_{t=1}^{\infty}$ converges. (steady state)

? Properties of steady state.

The Ramsey-Euler Equation:

$P(x)$ satisfies One-shot deviation property:

$$\forall x \geq 0: \quad P(x) \in \underset{\substack{y \in P(x) \\ P^2(x) \in P(y)}}{\operatorname{argmax}} \{ u(f(x) - y) + \delta u(f(y) - P^2(x)) \}$$

This yields Ramsey-Euler Equation:

$$u'(f(x) - P(x)) = \delta u'(f(P(x)) - P^2(x)) f'(P(x))$$

Steady state: $x^* = P(x^*)$.

$$\Rightarrow u'(f(\bar{x}) - \bar{x}) = \delta u'(f(\bar{x}) - \bar{x}) \cdot f'(\bar{x})$$

$$\Rightarrow f'(x) = \frac{1}{\delta}. \quad (\text{golden rule})$$

$f(\bar{x}) - \bar{x}$ is $\uparrow$ in δ (by Inada)

At steady state, no growth?

Exercise 4.3.

X, P, F, β sats. 4.1, 4.2.

let $v(x) = \sup_{y \in P(x)} [F(x,y) + \beta v(y)]$ $\quad v^* = \sup_{(x_t)_{t=1}^{\infty} \in \Omega(x_0)} \sum_{t=0} \beta^t F(x_t, x_{t+1})$

$\forall x_0 \in X, (x_t)_{t=1}^{\infty} \in \Omega(x_0)$, $\lim_{n\to\infty} \sup \beta^n v(x_n) \le 0$ $\quad u((x_t)) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$

① show that $v \le v^*$.

② Suppose in addition that $\forall x \in X$, $(x_t) \in \Omega(x_0)$, $\exists x' = (x'_t)_{t=1}^{\infty} \in \Omega(x_0)$

s.t. $\lim_{n\to\infty} \beta^n v(x'_n) = 0$ and $u((x'_t)) \ge u((x_t))$. show that $v = v^*$

$\forall x \in (x_t)_{t=1}^{\infty} \in \Omega(x_0)$, $u(x) = \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$

① WLOG, let $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) < \infty$.

Pick $(x_t)_{t=1}^{\infty} \in \Omega(x_0)$

$$\begin{aligned}
v(x_0) &\le F(x_0, x_1) + \beta v(x_1) + \varepsilon \\
&\le \sum_{t=0}^{T} \beta^t F(x_t, x_{t+1}) + \beta^{T+1} v(x_{T+1}) + \varepsilon \\
&\le \underline{\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})} + \limsup_{T\to\infty} \beta^{T+1} v(x_{T+1}) + \varepsilon \\
&\le \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) + \varepsilon \le \sup_{(x_t)_{t=1}^{\infty} \in \Omega(x_0)} \beta^t F(x_t, x_{t+1}) = v^*(x_0)
\end{aligned}$$

② $\sum_{t=0}^{\infty} \beta^t F(x'_t, x'_{t+1}) \ge \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$

$$\begin{aligned}
v(x_0) &\ge F(x_0, x'_1) + \beta v(x'_1) \\
&\ge \sum_{t=0}^{T} \beta^t F(x'_t, x'_{t+1}) + \underline{\beta^{T+1} v(x'_{T+1})}
\end{aligned}$$

$T \to \infty$ $\quad v(x_0) \ge \sum_{t=0}^{\infty} \beta^t F(x'_t, x'_{t+1}) \ge \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \quad \forall (x_t) \in \Omega(x_0)$

$\Rightarrow v(x_0) \ge \sup_{(x_t) \in \Omega(x_0)} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) = v^*(x_0)$

$\Rightarrow v(x_0) = v^*(x_0)$.

Thm 4.4: $X, \Gamma, F, \beta \sim$ Aspt 4.1, 4.2.

If $\exists\ (x_t^*) \in \Omega(x_0)$ s.t. $\sum_{t=0}^{\infty} \beta^t F(x_t^*, x_{t+1}^*) = \sup_{(x_t)\in\Omega(x_0)} \beta^t F(x_t, x_{t+1})$

Then $\forall t \in \mathbb{N}$ $v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + v^*(x_{t+1}^*)$

Pf: By induction. let $t=0$.

$$v^*(x_0) = F(x_0, x_1^*) + \sum_{t=1}^{\infty} \beta^t F(x_t^*, x_{t+1}^*)$$

$$\cancel{F(x_0, x_1^*)} + \sum_{t=1}^{\infty} \beta^t F(x_t^*, x_{t+1}^*) = \cancel{F(x_0, x_1^*)} + \sup_{(x_t)\in\Omega(x_0)} \beta^t F(x_t, x_{t+1})$$

since $x_1 \in \Gamma(x_0)$, $(x_t)_{t=2}^{\infty} \in \Omega(x_1)$.

$$\Rightarrow v^*(x_1^*) = \sup_{(x_t)\in\Omega(x_1)} \beta^t (x_t, x_{t+1})$$

$$\Rightarrow v^*(x_0) = F(x_0, x_1^*) + v^*(x_{t+1}^*).$$

suppose that $t=T$,

$$v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \sum_{T=t+1}^{\infty} \beta^T F(x_T^*, x_{T+1}^*).$$

Then $v^*(x_{t+1}^*) =$